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Linearity in Deng entropy

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ABSTRACT

Linearity is considered as a fundamental property of significant importance in numerous domains. A linear mathematical expression is clear, concise, and convenient for solving problems, making it a valuable tool for approximating complex issues. In recent years, Deng entropy has been proposed as a generalization of Shannon entropy, applied to measure the uncertainty degree of the mass function in the power set. There are two main contributions in this paper. One is that the linearity can be observed in Deng entropy. We present a set of specific mass functions named power assigned mass function (PAMF), which could generate linear type Deng entropy (LTDE). The other is that we find the slope is nothing else but the information fractal dimension of mass function. Moreover, we discover that for any given slope within the range of 0 to $\frac{\ln 3}{\ln 2}$, at least one mass function that yields Deng entropy corresponding to the given slope can be derived through the mass function generator (MFG), in a strict linear way. Some proofs, numerical examples, discussions and an error analysis are provided to validate the effectiveness of our findings.

1. Introduction

The measurement of uncertainty has long been a pivotal research focus in the information theory. In order to model and manage information with uncertainty, various related theories have been developed, such as probability theory [1], Dempster-Shafer evidence theory [2,3], fuzzy sets [4], quantum evidence theory [5,6], and random permutation set [7]. These theories have found extensive applications across diverse domains including risk analysis [8], decision making [9], pattern classification [10–13], cellular automaton [14,15], fractal theory [16–19], information fusion [20], complex networks [21,22], information measure [23] and information dimension [24,25].

In the information theory, entropy is a powerful tool for quantifying the uncertainty of information. Various types of information entropy have been proposed, such as Shannon entropy [26], Rényi entropy [27], Tsallis entropy [28], fuzzy entropy [29], complex entropy [30], and quantum X-entropy [6] have found specific applications [31–33]. Deng entropy [34], as a generalization of Shannon entropy, effectively measures the uncertainty degree of the mass function in the power set. In recent years, extensive studies have been devoted to exploring its properties [35,36], generalizations [37], and maximum entropy [38]. Furthermore, Deng entropy has found applications in a multitude of fields, including probability distribution [39, 40], information volume [41–43], decision making [44,45], feature extraction [46], cellular automaton [47] and time series analysis [48, 49].

The linearity property represents a fundamental and essential characteristic that finds widespread application in various research domains, including linear regression in statistics [50], linear quadratic regulator (LQR) in control theory [51], linear systems in signal processing [52], linear fractal interpolation function in fractal interpolation [53,54] and support vector machine (SVM) in machine learning and deep learning [55,56]. Interestingly, linearity can be observed in studies related to Deng entropy as well. In the numerical examples presented in [34], we distinctly observe that Deng entropy corresponding to certain mass functions exhibits a linear trend concerning the scale of the frame of discernment (SFOD). Once again, in the study of the maximum Deng entropy [38], a strong linear relationship is presented between the maximum Deng entropy and SFOD. Last but not least, with the proposal of the information dimension of mass function based on Deng entropy [57], the linearity in Deng entropy becomes increasingly apparent. Currently, there are three mass functions that result in linear type Deng entropy (LTDE): the first one is mass function corresponding to the maximum Deng entropy ($m(A) = \frac{2^{|A|}-1}{\sum_{B\subseteq\Theta} 2^{|B|}-1}$), with a slope equal

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to $\log_2(3)$; the second one is mass function with average distribution ($m(A) = \frac{1}{2^N - 1}$), with a corresponding slope of 1.5; the third one is mass function with total uncertainty ($m(\Theta) = 1$), with a slope equal to 1 [57].

The presence of linearity in Deng entropy has raised two significant questions: (1) Are there other mass functions that could result in a linear relationship between Deng entropy and SFOD? (2) Given a specific slope, can we find mass functions that yield Deng entropy with that slope? Inspired by these questions, this paper conducts an in-depth exploration of the linear relationship between Deng entropy and SFOD. The principal contributions of this paper include the discovery of a set of specific mass functions named power assigned mass function (PAMF) which could generate linear type Deng entropy (LTDE). The slope of Deng entropy is nothing else but the information fractal dimension of mass function. More importantly, we find that for any given slope within the range of 0 to $\frac{\ln 3}{\ln 2}$, at least one mass function that yields Deng entropy corresponding to the given slope can be derived through the mass function generator (MFG). Furthermore, we also provide several proofs, numerical examples, discussions and an error analysis to validate the effectiveness of our findings.

The structure of the remaining sections is outlined as follows. In Section 2, we will introduce some preliminary concepts relevant to this study. In Section 3, we will explore the linearity in Deng entropy based on the two questions posed earlier. Section 4 will provide some numerical examples to validate the effectiveness of our findings. Section 5 will conduct an analysis of the errors and engage in some discussions. Finally, in Section 6, we will offer a comprehensive summary of the entire article.

2. Preliminaries

In this section, the concepts of mass function, Deng entropy, the maximum Deng entropy and information fractal dimension of mass function are briefly introduced. We also conduct an initial exploration of the linearity observed in Deng entropy.

2.1. Mass function

Mass function is a foundational concept in Dempster-Shafer evidence theory [2,3], comparable to the probability mass function in the probability theory [1]. The distinguishing feature lies in the domain of definition: the mass function is defined over the power set, while the probability mass function is defined over the sample space.

Definition 2.1 (*Frame of Discernment, FOD*)**.** A finite set $\Theta = \{\theta_1, \theta_2, \dots, \theta_N\}$ which contains N mutually exclusive elements is defined as the frame of discernment (FOD). The cardinality of the frame of discernment $|\Theta|$ ($|\Theta| = N$) is called the scale of the frame of discernment (SFOD). The family of sets $2^\Theta = \{\Phi, \{\theta_1\}, \dots, \{\theta_N\}\, \{\theta_1, \theta_2\}, \dots, \Theta\}$ that consists of all subsets of Θ (including Θ itself and the empty set Φ) is defined as the power set of Θ.

Definition 2.2 (*Mass Function*)**.** For a given FOD Θ, a mass function is defined as a mapping function m from 2^Θ to $[0, 1]$:

$$m : 2^\Theta \to [0, 1] \tag{1}$$

which is constrained by:

$$m(\Phi) = 0 \quad \text{and} \quad \sum_{A \in 2^\Theta} m(A) = 1 \tag{2}$$

If $m(A) > 0$, A is defined as a focal element of m. A mass function can also be called a basic probability assignment (BPA).

2.2. Deng entropy

To measure the uncertainty of mass functions in the power set, Deng proposed a novel entropy named Deng entropy [34] (also called belief entropy) in 2016. Furthermore, Kang and Deng studied the condition and the analytic solution of the maximum Deng entropy [38] in 2019.

Definition 2.3 (*Deng Entropy*)**.** For a given mass function m defined on the FOD $\Theta = \{\theta_1, \theta_2, \dots, \theta_N\}$, Deng entropy is defined as follows:

$$E_D(m) = -\sum_{A \in 2^\Theta} m(A) \cdot \log_2 \left[\frac{m(A)}{2^{|A|} - 1} \right] \tag{3}$$

where A is the focal element.

Theorem 2.1 (*The Condition of the Maximum Deng Entropy*)**.** *For a given mass function m defined on the FOD $\Theta = \{\theta_1, \theta_2, \dots, \theta_N\}$, the maximum Deng entropy is derived if and only if the mass function m meets the following condition:*

$$m(A) = \frac{2^{|A|} - 1}{\sum_{B \in 2^\Theta} \left(2^{|B|} - 1\right)}, \ \forall A \subseteq \Theta \tag{4}$$

Theorem 2.2 (*The Analytic Solution of the Maximum Deng Entropy*)**.** *For a given FOD $\Theta = \{\theta_1, \theta_2, \dots, \theta_N\}$, the maximum Deng entropy can be derived through the analytic solution, which is expressed as:*

$$E_{Dmax} = \log_2 \left[\sum_{F \in 2^\Theta} \left(2^{|F|} - 1\right) \right] \tag{5}$$

2.3. Information fractal dimension of mass function

Definition 2.4 (*Information Fractal Dimension of Mass Function*)**.** For a given mass function m defined on the FOD $\Theta = \{\theta_1, \theta_2, \dots, \theta_N\}$, its information dimension is defined as follows [25,57]:

$$D_m = \frac{E_D}{\log \left[\sum_{X \in 2^\Theta} \left(2^{|X|} - 1\right)^{m(X)} \right]} \tag{6}$$

where E_D is the Deng entropy of the given mass function.

2.4. Some examples that show linearity in Deng entropy

In the last part of this section, we will present some certain mass functions that show linearity in Deng entropy. Assuming a variable-scale FOD Θ_n, m_a is the mass function corresponding to the maximum Deng entropy; m_b is the mass function with average distribution; m_c is the mass function with total uncertainty. The specific expressions for their mass functions are as follows:

$$m_a(F) = \frac{2^{|F|} - 1}{\sum_{B \subseteq \Theta_n} 2^{|B|} - 1}, \ \forall F \subseteq \Theta_n \tag{7}$$

$$m_b(F) = \frac{1}{2^{|\Theta_n|} - 1}, \ \text{for} \ \ F \neq \Phi \tag{8}$$

$$m_c(\Theta_n) = 1 \tag{9}$$

The relevant numerical values are presented in Table 1, where E_a represents the Deng entropy derived from mass function m_a, while $\text{Diff}_a = E_a^{(n)} - E_a^{(n-1)}$ (here $E_a^{(n)}$ denotes the Deng entropy derived from mass function m_a when $|\Theta_n| = n$. In particular, when $n = 0$, let $E_a^{(0)} = 0$). The same definitions apply to b and c. Moreover, the Entropy-SFOD plot and the Difference-SFOD plot of these mass functions are illustrated in Figs. 1 and 2. The x-axis represents the scale of the frame of discernment (SFOD), while the y-axis represents Deng entropy or difference, i.e. $E^{(n)} - E^{(n-1)}$. An observable trend is that as the SFOD increases, the difference gradually converges to a constant value. It suggests that the Deng entropy corresponding to these three mass functions exhibits a linear relationship with SFOD, and the value to which $E^{(n)} - E^{(n-1)}$ converges corresponds to the slope. It is interesting

Table 1
The relevant numerical values of the mass functions.

$\|\Theta_n\|$	m_a		m_b		m_c	
	E_a	Diff_a	E_b	Diff_b	E_c	Diff_c
1	0	0	0	0	0	0
2	2.32193	2.32193	2.11328	2.11328	1.58496	1.584961
3	4.24793	1.92600	3.88768	1.77439	2.80735	1.22239
4	6.02237	1.77444	5.54996	1.66229	3.90689	1.09954
5	7.72110	1.69873	7.16103	1.61107	4.95420	1.04731
6	9.37721	1.65611	8.74279	1.58176	5.97728	1.02308
⋮	⋮	⋮	⋮	⋮	⋮	⋮
48	76.07820	1.58496	70.50000	1.50000	49.00000	1.00000
49	77.66316	1.58496	72.00000	1.50000	50.00000	1.00000
50	79.24813	1.58496	73.50000	1.50000	51.00000	1.00000
⋮	⋮	⋮	⋮	⋮	⋮	⋮
98	155.32633	1.58496	147.00000	1.50000	98.00000	1.00000
99	156.91129	1.58496	148.50000	1.50000	99.00000	1.00000
100	158.49625	1.58496	150.00000	1.50000	100.00000	1.00000

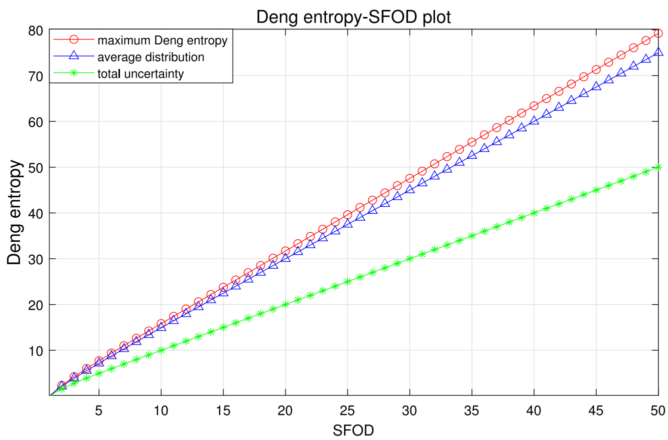

Fig. 1. The Deng entropy-SFOD plot of these three mass functions (please note that the *x*-axis coordinates start from 1).

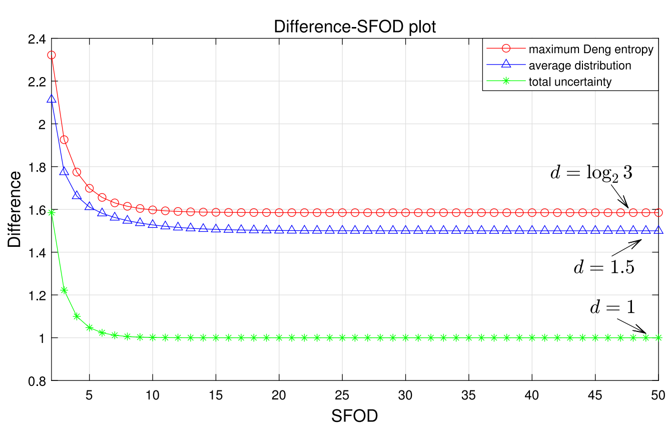

Fig. 2. The Difference-SFOD plot of these three mass functions.

that the slopes are equal to the information fractal dimensions of these mass functions [57]. Therefore, the slope of Deng entropy is nothing else but the information fractal dimension of mass function.

In this part, we emphasize illustrating the linearity in Deng entropy intuitively. In Fig. 1, a clear linear pattern is observed. Additionally, in Table 1 and Fig. 2, the differences between the y-values of adjacent points on the same line in Fig. 1 are computed. Each of the differences reveals a gradual convergence to a constant value. Whether the convergence is transient or enduring, and whether this linearity is a fortuitous occurrence or an inherent property will be discussed more comprehensively in Sections 3 and 4.

3. Explore the linearity in deng entropy

In this section, we will conduct an in-depth exploration of the linearity in Deng entropy. Let us now review the two questions we raised earlier: (1) Are there other mass functions that could result in a linear relationship between Deng entropy and SFOD? (2) Given a specific slope, can we find mass functions that yield Deng entropy with that slope? Firstly, LTDE should be defined.

Definition 3.1 (*Linear Type Deng Entropy, LTDE*)**.** Given a variable-scale FOD $\Theta_n = \{\theta_1, \theta_2, \dots, \theta_n\}$ and a mass function m defined on that FOD (variable-scale means the SFOD $|\Theta_n| = n$ is a variable), the Deng entropy corresponding to the mass function m is E_D. As $n \to \infty$, if $\frac{E_D}{n}$ approaches a positive constant value, then the Deng entropy E_D is defined as a linear type Deng entropy (LTDE).

Definition 3.1 means that: $\exists\ k > 0,\ \forall\ \varepsilon > 0,\ \exists\ N \in Z+$, if $n > N$, $|\frac{E_D}{n} - k| < \varepsilon$. Next, let us proceed from the analytic solution of the maximum Deng entropy:

$$
\begin{aligned}
E_{Dmax} &= \log_2\left[\sum_{F\in 2^{\Theta}} \left(2^{|F|}-1\right)\right] \\
&= \log_2\left\{\sum_{k=0}^{|\Theta|}\left[\binom{|\Theta|}{k}\cdot\left(2^k-1\right)\right]\right\} \\
&= \log_2\left\{\sum_{k=0}^{|\Theta|}\left[\binom{|\Theta|}{k}\cdot 2^k\right]-\sum_{k=0}^{|\Theta|}\binom{|\Theta|}{k}\right\} \\
&= \log_2\left(3^{|\Theta|}-2^{|\Theta|}\right)
\end{aligned}
\tag{10}
$$

where $|\Theta|$ denotes the cardinality of the FOD Θ, $\binom{|\Theta|}{k}$ represents the combinatorial number, calculated using the formula $\binom{|\Theta|}{k} = \frac{|\Theta|!}{k!\cdot(|\Theta|-k)!}$. In the final step of the calculation in Eq. (10), the classic binomial theorem [58] is applied for simplification.

When $|\Theta| \to \infty$, $3^{|\Theta|}$ is much larger than $2^{|\Theta|}$. Therefore, in Eq. (10), we can disregard $2^{|\Theta|}$ and approximate the maximum Deng entropy using the following expression:

$$
\begin{aligned}
E_{Dmax} &= \log_2\left(3^{|\Theta|}-2^{|\Theta|}\right) \\
&\approx \log_2\left(3^{|\Theta|}\right) \\
&= \log_2(3)\cdot|\Theta|
\end{aligned}
\tag{11}
$$

In Eq. (11), as $|\Theta| \to \infty$, we finally discover the linear relationship (proportional relationship) between the maximum Deng entropy E_{Dmax} and SFOD $|\Theta|$, with the slope (proportionality factor) being $\log_2(3)$, which is consistent with experimental results in Figs. 1 and 2. Furthermore, we can use L'Hôpital's rule to demonstrate that as $|\Theta| = N \to \infty$, the ratio of E_{Dmax} to N approaches $\log_2(3)$ (See Proof 1).

Proof 1. (Proof for the Ratio $\log_2(3)$, $N = |\Theta|$.)

$$
\begin{aligned}
\lim_{N\to\infty}\frac{E_{Dmax}}{N} &= \lim_{N\to\infty}\frac{\log_2\left(3^N-2^N\right)}{N} \\
&\overset{\frac{\infty}{\infty}}{=} \lim_{N\to\infty}\frac{\frac{3^N\cdot\ln 3-2^N\cdot\ln 2}{(3^N-2^N)\cdot\ln 2}}{1} \\
&= \lim_{N\to\infty}\frac{\ln 3-(\frac{2}{3})^N\cdot\ln 2}{\ln 2-(\frac{2}{3})^N\cdot\ln 2} \\
&= \frac{\ln 3}{\ln 2} \\
&= \log_2(3)
\end{aligned}
\tag{12}
$$

Inspired by the reasoning process above, we consider whether mass functions similar to Eq. (4) can also yield linear type Deng entropy (LTDE). This idea paves the way for our next research part.

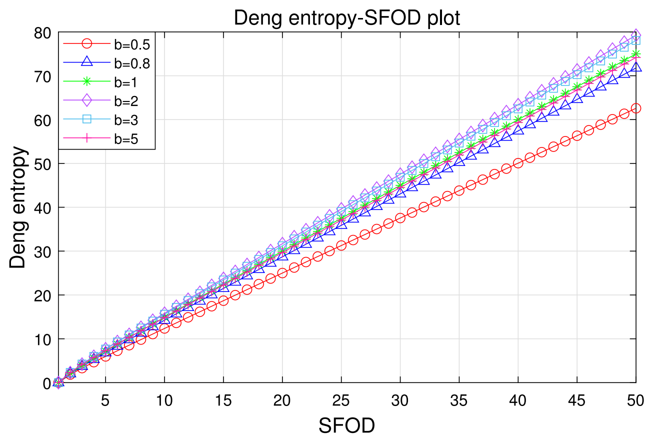

Fig. 3. Deng entropy-SFOD plot for PAMF corresponding to different values of b.

3.1. PAMF: new mass functions that result in LTDE

Following the format of Eq. (4), we will provide the definition of power assigned mass function (PAMF).

Definition 3.2 (*Power Assigned Mass Function, PAMF*)**.** Given a FOD $\Theta = \{\theta_1, \theta_2, \ldots, \theta_N\}$, for any non-empty subset $A \subseteq \Theta$, its corresponding power assigned mass function (PAMF) is as follows:

$$m_P(A) = \frac{b^{|A|}}{\sum_{F\subseteq\Theta} b^{|F|}} \tag{13}$$

where b is a positive constant (real number) named power assigned coefficient (PAC), F is the non-empty subset, $|A|$ and $|F|$ denote the cardinality of A and F. It means distributing $b^{|A|}$ to each non-empty subset $A \subseteq \Theta$, and the sum is the accumulation of these distributions across all subsets.

Fig. 3 presents an Entropy-SFOD plot for mass functions corresponding to different values of b. It is evident that these curves all exhibit strong linearity. In the following part, we will demonstrate the rationality of this linear phenomenon.

Theorem 3.1. *For a given mass function m defined on the FOD $\Theta = \{\theta_1, \theta_2, \ldots, \theta_N\}$, linear type Deng entropy (LTDE) can be derived if the mass function m is a power assigned mass function (PAMF).*

Proof 2. (Proof for Theorem 3.1). The PAMF in Eq. (13) can be rewritten as:

$$m_P(A) = \frac{b^a}{\sum_{k=1}^{N}\binom{N}{k}\cdot b^k} = \frac{b^a}{(b+1)^N - 1} \tag{14}$$

where a and N is the cardinality of A and the FOD, $\binom{N}{k} = \frac{N!}{k!\cdot(N-k)!}$. Substituting the mass function from Eq. (14) into the Deng entropy expression in Eq. (3):

$$\begin{aligned}
E_D(m_P) &= -\sum_{A\in 2^\Theta} m_P(A)\cdot \log_2\left[\frac{m_P(A)}{2^{|A|}-1}\right] \\
&= -\sum_{k=1}^{N}\binom{N}{k}\cdot\frac{b^k}{(b+1)^N-1}\log_2\left[\frac{\frac{b^k}{(b+1)^N-1}}{2^k-1}\right] \\
&= \frac{\sum_{k=1}^{N}\left\{\binom{N}{k}\cdot b^k\cdot\log_2\left[(b+1)^N-1\right] - \binom{N}{k}\cdot b^k\cdot\log_2\left(\frac{b^k}{2^k-1}\right)\right\}}{(b+1)^N-1} \\
&= \log_2\left[(b+1)^N-1\right] - \frac{\sum_{k=1}^{N}\left[\binom{N}{k}\cdot b^k\cdot\log_2\left(\frac{b^k}{2^k-1}\right)\right]}{(b+1)^N-1}
\end{aligned} \tag{15}$$

when N is sufficiently large, the '-1' term in Eq. (15) can be neglected, and thus it can be rewritten as:

$$E_D(m_P) \approx N\cdot\log_2(b+1) - \frac{\sum_{k=1}^{N}\left[\binom{N}{k}\cdot b^k\cdot k\cdot\log_2\left(\frac{b}{2}\right)\right]}{(b+1)^N} \tag{16}$$

By comparing Eq. (15) with Eq. (16), it is evident that after approximation, E_D in Eq. (16) will be greater than that in Eq. (15) when $b > 2$. In Section 5, we will conduct further analysis of the error introduced during the approximation process.

By applying the binomial theorem [58], we can derive the following expression:

$$(x+1)^N = \sum_{k=1}^{N}\left[\binom{N}{k}\cdot x^k\right] + 1 \tag{17}$$

then take the derivative of both sides of Eq. (17) with respect to the variable x:

$$N\cdot(x+1)^{N-1} = \sum_{k=1}^{N}\left[\binom{N}{k}\cdot k\cdot x^{k-1}\right] \tag{18}$$

next, multiply both sides of Eq. (18) by x:

$$x\cdot N\cdot(x+1)^{N-1} = \sum_{k=1}^{N}\left[\binom{N}{k}\cdot k\cdot x^{k}\right] \tag{19}$$

according to Eq. (19), Eq. (16) can be rewritten as:

$$\begin{aligned}
E_D(m_P) &= N\cdot\log_2(b+1) - \frac{\log_2\left(\frac{b}{2}\right)\cdot N\cdot b\cdot(b+1)^{N-1}}{(b+1)^N} \\
&= N\cdot\log_2(b+1) - \log_2\left(\frac{b}{2}\right)\cdot N\cdot\frac{b}{b+1} \\
&= \left[\log_2(b+1) - \log_2\left(\frac{b}{2}\right)\cdot\frac{b}{b+1}\right]\cdot N
\end{aligned} \tag{20}$$

where N is the cardinality of the FOD, i.e. the SFOD $|\Theta|$. According to Eq. (20), the linear relationship between Deng entropy and SFOD is evident. So it is proved that linear type Deng entropy can be derived if the mass function m is a power assigned mass function. The proof for Theorem 3.1 is completed.

3.2. MFG: generating mass functions from a given slope

From Eq. (20), we can observe that the slope of the Deng entropy derived from the PAMF is exclusively determined by the power assigned coefficient b. This implies that for a given slope of the LTDE, we can inversely deduce the value of b, and further derive the PAMF.

Definition 3.3 (*Mass Function Generator, MFG*)**.** For a real number $b \in (0, +\infty)$, the mass function generator (MFG) is defined by the following function:

$$G(b) = \log_2(b+1) - \log_2\left(\frac{b}{2}\right)\cdot\frac{b}{b+1} \tag{21}$$

where b is the power assigned coefficient (PAC), $G(b)$ could be the given slope. Fig. 4 shows the Slope-PAC plot of the MFG when $b \in (0, 20)$.

Theorem 3.2. *Given a linear type Deng entropy (LTDE) with a certain slope k, for any $k \in (0, \log_2 3]$, at least one power assigned mass function (PAMF) that yields Deng entropy corresponding to the slope can be derived through the mass function generator (MFG). Furthermore:*

- *If $k \in (0, 1]$ or $k = \log_2(3)$, one power assigned mass function can be derived;*
- *If $k \in (1, \log_2 3)$, two power assigned mass functions can be derived.*

Proof 3. (Proof for Theorem 3.2). From Eqs. (20) and (21) we can know:

$$k = G(b) = \log_2(b+1) - \log_2\left(\frac{b}{2}\right)\cdot\frac{b}{b+1} \tag{22}$$

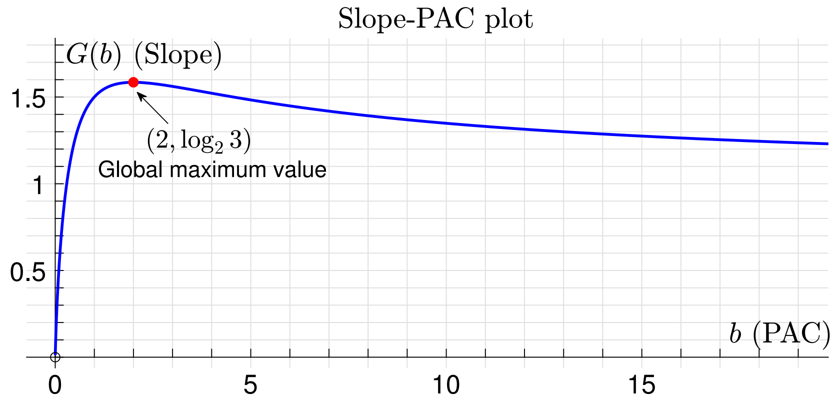

Fig. 4. The Slope-PAC plot of the MFG when $b \in (0, 20)$.

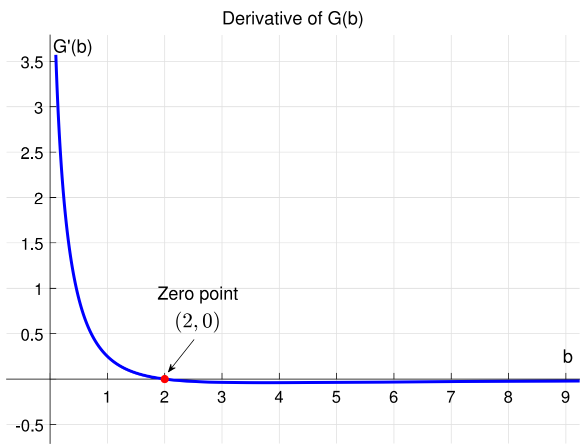

Fig. 5. The plot of $G'(b)$.

Eq. (22) can be rewritten as:

$$G(b) = \log_2(b+1) - \log_2\left(\frac{b}{2}\right) + \frac{1}{b+1}\cdot\log_2\left(\frac{b}{2}\right) \tag{23}$$

then take the derivative of $G(b)$ with respect to b:

$$\begin{aligned} G'(b) &= \frac{\mathrm{d}G(b)}{\mathrm{d}b} \\ &= \frac{1}{(b+1)\cdot\ln 2} - \frac{1}{b\cdot\ln 2} - \frac{1}{(b+1)^2}\cdot\log_2\left(\frac{b}{2}\right) + \frac{1}{b+1}\cdot\frac{1}{b\cdot\ln 2} \\ &= \frac{b-(b+1)+1}{b\cdot(b+1)\cdot\ln 2} - \frac{1}{(b+1)^2}\cdot\log_2\left(\frac{b}{2}\right) \\ &= -\log_2\left(\frac{b}{2}\right)\cdot\frac{1}{(b+1)^2} \end{aligned} \tag{24}$$

The plot of $G'(b)$ is shown in Fig. 5. Combining Fig. 5 with Eq. (24) provides the following information:

- When $b \in (0, 2)$, $G'(b) > 0$;
- When $b = 2$, $G'(b) = 0$;
- When $b \in (2, +\infty)$, $G'(b) < 0$;

Further deductions can be made based on the properties of the derivative:

- When $b \in (0, 2)$, $G(b)$ is monotonically increasing;
- When $b = 2$, $G(b)$ achieves its global maximum value $\log_2(3)$;
- When $b \in (2, +\infty)$, $G(b)$ is monotonically decreasing.

Moreover:

$$\begin{aligned} \lim_{b\to+\infty} G(b) &= \lim_{b\to+\infty} \log_2(b+1) - \log_2\left(\frac{b}{2}\right) \\ &= \lim_{b\to+\infty} \log_2(b+1) - \log_2(b) + 1 \\ &= 1 \end{aligned} \tag{25}$$

$$\begin{aligned} \lim_{b\to0^+} G(b) &= \lim_{b\to0^+} \log_2\left(\frac{b}{2}\right)\cdot\frac{-b}{b+1} \\ &= \lim_{b\to0^+} \frac{-\log_2\left(\frac{b}{2}\right)}{1+\frac{1}{b}} \\ &\overset{\frac{\infty}{\infty}}{=} \lim_{b\to0^+} \frac{\frac{1}{b\cdot\ln 2}}{\frac{1}{b^2}} \\ &= \lim_{b\to0^+} \frac{b}{\ln 2} \\ &= 0 \end{aligned} \tag{26}$$

where $b \to 0^+$ implies that b approaches 0 from the positive side. When $b \in (0, 1000)$, the plot of $G(b)$ is shown in Fig. 6. Therefore, as inferred from Fig. 6 and Eqs. (22), (25), (26), for a certain slope $k \in (0, \log_2 3]$, when inputting k into the MFG $G(b)$, assuming $G(C) = 1$ (C is a constant, the value of C will be discussed in Example 4.2), the following conclusions can be drawn:

- If $k \in (0, 1]$ or $k = \log_2(3)$, then it is possible to obtain a value of $b \in (0, C]$ or $b = 2$ such that $G(b) = k$;
- If $k \in (1, \log_2 3)$, then values of $b_1 \in (C, 2)$ and $b_2 \in (2, +\infty)$ can be found such that $G(b_1) = G(b_2) = k$.

The proof for Theorem 3.2 is completed. This theorem is intuitive because when $b \to \infty$, the PAMF tends to make $m(\Theta) = 1$, corresponding to the situation of total uncertainty. In this scenario, the slope k of the LTDE approaches 1, consistent with Fig. 2. Conversely, when b is small, the PAMF tends to allocate mass to individual subsets $(\{\theta_1\}, \{\theta_2\}, \ldots, \{\theta_N\})$, resulting in lower uncertainty, and hence, the LTDE has a smaller slope. Moreover, when $b = 2$, the slope reaches its global maximum value of $k = \log_2(3)$, corresponding to the situation of maximum Deng entropy, in line with Fig. 2 and Eq. (11).

The convexity property of a function is also a significant mathematical property [23,49]. We can examine the convexity property of $G(b)$ by checking the sign of its second derivative:

$$\begin{aligned} G''(b) &= \frac{\mathrm{d}^2G(b)}{\mathrm{d}b^2} \\ &= \frac{\mathrm{d}G'(b)}{\mathrm{d}b} \\ &= \frac{2}{(b+1)^3}\cdot\log_2\left(\frac{b}{2}\right) - \frac{1}{b\cdot\ln 2}\cdot\frac{1}{(b+1)^2} \\ &= \frac{2\cdot\ln\left(\frac{b}{2}\right)}{\ln 2\cdot(b+1)^3} - \frac{1}{\ln 2\cdot b\cdot(b+1)^2} \\ &= \frac{2b\cdot\ln\left(\frac{b}{2}\right)-(b+1)}{\ln 2\cdot b\cdot(b+1)^3} \end{aligned} \tag{27}$$

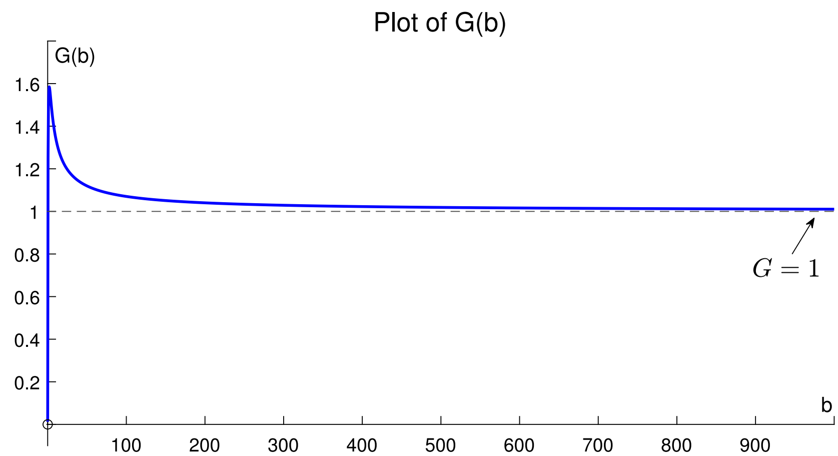

Fig. 6. The plot of $G(b)$ when $b \in (0, 1000)$.

Since b is a positive number, the denominator of Eq. (27) is greater than 0. Taking the numerator separately, define the function $f(b)$ as:

$$f(b) = 2b \cdot \ln\left(\frac{b}{2}\right) - b - 1 \tag{28}$$

then take the derivative of $f(b)$ with respect to b:

$$\begin{aligned} f'(b) &= \frac{\mathrm{d}f(b)}{\mathrm{d}b} \\ &= 2 \cdot \ln\left(\frac{b}{2}\right) + 2b \cdot \frac{1}{b} - 1 \\ &= 2 \cdot \ln\left(\frac{b}{2}\right) + 1 \end{aligned} \tag{29}$$

In Eq. (29), it is obvious that $f'(b)$ is monotonically increasing within the interval $b \in (0, +\infty)$, having a unique zero point at $b_0' = 2e^{-\frac{1}{2}}$. Furthermore, the analysis reveals that the function $f(b)$ is monotonically decreasing in the interval $b \in (0, 2e^{-\frac{1}{2}})$ and monotonically increasing in the interval $b \in (2e^{-\frac{1}{2}}, +\infty)$. When $b \to 0^+$, it can be calculated that $\lim_{b\to 0^+} = -1$ by using L'Hôpital's rule. Let $f(b) = 0$, the unique zero point of $f(b)$ can be solved as $(3.7657, 0)$ ($b_0 \approx 3.7656697\ldots$). The plot of $f(b)$ is shown in Fig. 7.

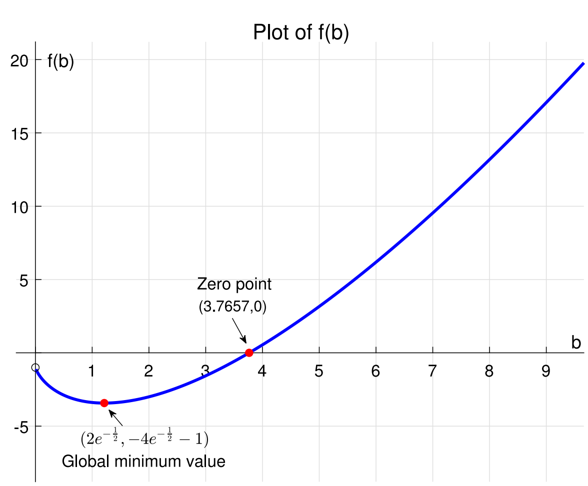

Fig. 7. The plot of $f(b)$.

Therefore, the plot of $G''(b)$ is shown in Fig. 8 and the following conclusions can be drawn:

- When $b \in (0, 3.7657)$, $G''(b) < 0$, indicating that $G(b)$ is concave with respect to b;
- When $b \in (3.7657, +\infty)$, $G''(b) > 0$, indicating that $G(b)$ is convex with respect to b.

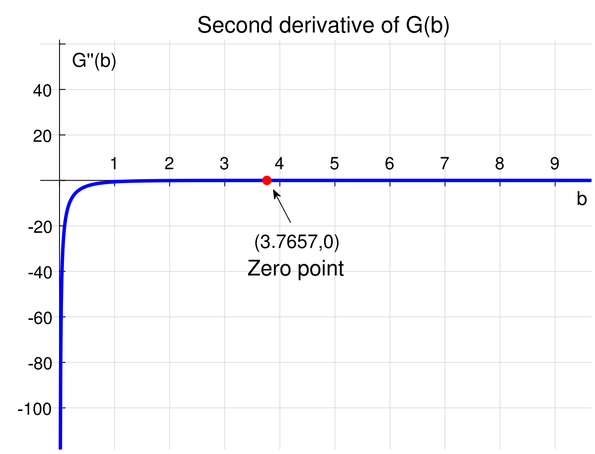

Fig. 8. The plot of $G''(b)$.

4. Numerical examples

Example 4.1. *Given a PAMF with the PAC $b = 1$, solve the slope k of the LTDE corresponding to that PAMF.*

By using Definition 3.2, the PAMF could be expressed as:

$$m_P = \frac{1}{2^N - 1} \tag{30}$$

Obviously, Eq. (30) is the situation of average distribution. According to Eq. (20), let $b = 1$, then:

$$k = \log_2(1+1) - \log_2\left(\frac{1}{2}\right) \cdot \frac{1}{1+1} = 1.5 \tag{31}$$

This result is consistent with Fig. 2. According to Theorem 3.2, since $k = 1.5 \in (1, \log_2 3)$, there should exist another PAMF with the PAC $b' \in (2, +\infty)$, such that the slope k of the LTDE corresponding to that PAMF is also 1.5. b' could be derived by inputting $k = 1.5$ into the MFG:

$$\log_2(b+1) - \log_2\left(\frac{b}{2}\right) \cdot \frac{b}{b+1} = 1.5 \tag{32}$$

The solution to Eq. (32) is: $b_1 = 1$, $b_2 = 4.5575580$. Use the PAMF with the PAC $b = 4.5575580$, calculate its corresponding Deng entropy. The Entropy-SFOD plot for that PAMF is shown in Fig. 9. It is obvious that the Deng entropy and the SFOD exhibit a strong linear relationship. The slope of the Deng entropy is 1.5, which is in line with theoretical calculation.

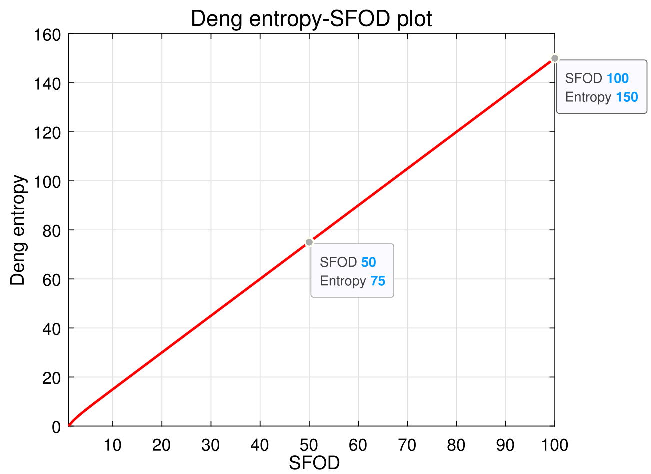

Fig. 9. The Entropy-SFOD plot for the PAMF with $b = 4.5575580$ (please note that the x-axis coordinates start from 1).

Example 4.2. *Given a slope $k = 1$, derive the mass function(s) which could yield Deng entropy corresponding to that slope.*

According to Theorem 3.2, since $k = 1 \in (0, 1]$, there should exist one PAMF. Input $k = 1$ into the MFG:

$$\log_2(b+1) - \log_2\left(\frac{b}{2}\right) \cdot \frac{b}{b+1} = 1 \tag{33}$$

The solution to Eq. (33) is: $b = 0.2938154$. The Entropy-SFOD plot for the PAMF with $b = 0.2938154$ is shown in Fig. 10. Therefore, the value of the constant C in Proof 3 is 0.2938154.

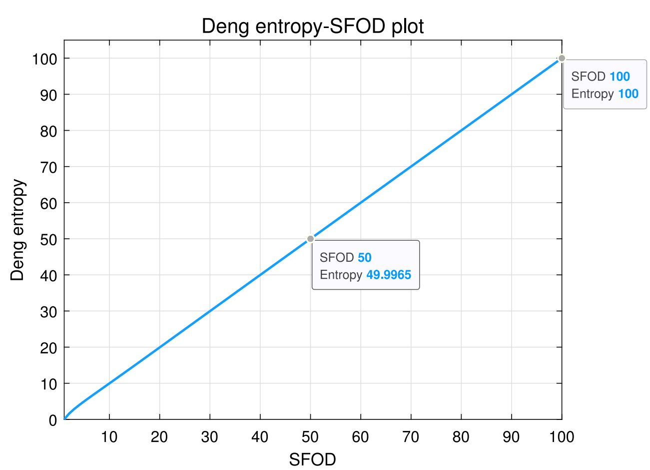

Fig. 10. The Entropy-SFOD plot for the PAMF with $b = 0.2938154$.

Example 4.3. *Given a PAMF with the PAC $b = 10$, solve the slope k of the LTDE corresponding to that PAMF.*

Let $b = 10$ in Eq. (20):

$$k = \log_2(10+1) - \log_2\left(\frac{10}{2}\right) \cdot \frac{10}{10+1} = 1.3485879 \tag{34}$$

The Entropy-SFOD plot for the PAMF with $b = 10$ is shown in Fig. 11. Similarly, another $b = 0.6293296$ which leads to the LTDE with slope $k = 1.3485879$ could be derived through the MFG in Eq. (21). The result is consistent with theoretical calculation, Theorem 3.1 and Theorem 3.2.

Example 4.4. *Given a slope $k = \ln 2$, derive the mass function(s) which could yield Deng entropy corresponding to that slope.*

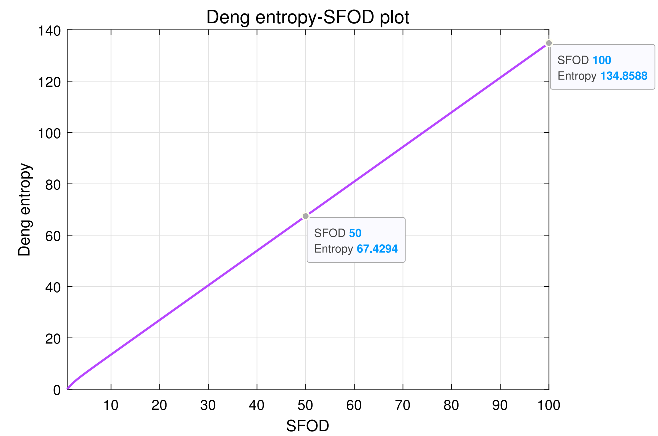

Fig. 11. The Entropy-SFOD plot for the PAMF with $b = 10$.

According to Theorem 3.2, since $k = \ln 2 \in (0, 1]$, there should exist one PAMF with PAC $b \in (0, 0.2938154]$. Input $k = \ln 2$ into the MFG:

$$\log_2(b+1) - \log_2\left(\frac{b}{2}\right) \cdot \frac{b}{b+1} = \ln 2 \tag{35}$$

The solution to Eq. (33) is: $b = 0.1514495$. The Entropy-SFOD plot for the PAMF with $b = 0.1514495$ is shown in Fig. 12. It can be observed that in this example, when SFOD = 100, there is a slight deviation between the calculated slope and the given slope. This deviation may be attributed to the SFOD not being sufficiently large because the deviation is reduced as the SFOD increases to 150. Further analysis regarding the error analysis will be presented in Section 5.

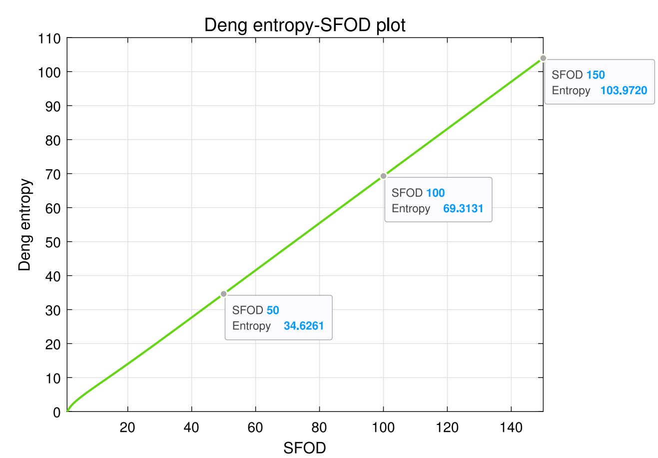

Fig. 12. The Entropy-SFOD plot for the PAMF with $b = 0.1514495$.

5. Error analysis and discussion

Analyzing the provided numerical examples, the errors in approximating Deng entropy with a straight line passing through the origin primarily originate from the approximation in Eq. (16) and the errors introduced by software calculations. The errors are more pronounced when SFOD is small and may exhibit some variability. Nevertheless, as SFOD increases incrementally, the errors will diminish. With a sufficiently large SFOD, the errors become negligible and can be safely disregarded.

We select some of the examples in Section 4 and plotted the LTDE along with its corresponding fitted straight lines, as shown in Fig. 13. In this figure, the y-coordinates of the scattered points represent the Deng entropy generated by PAMF with different PAC b, while the dashed lines are straight lines passing through the origin with different slope

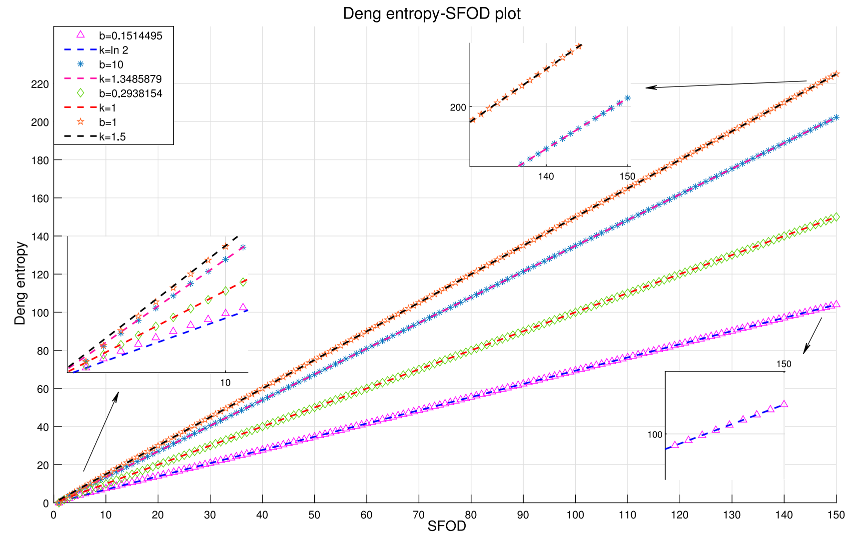

Fig. 13. The Entropy-SFOD plot for the PAMFs with different PAC b and straight lines passing through the origin with different slope k.

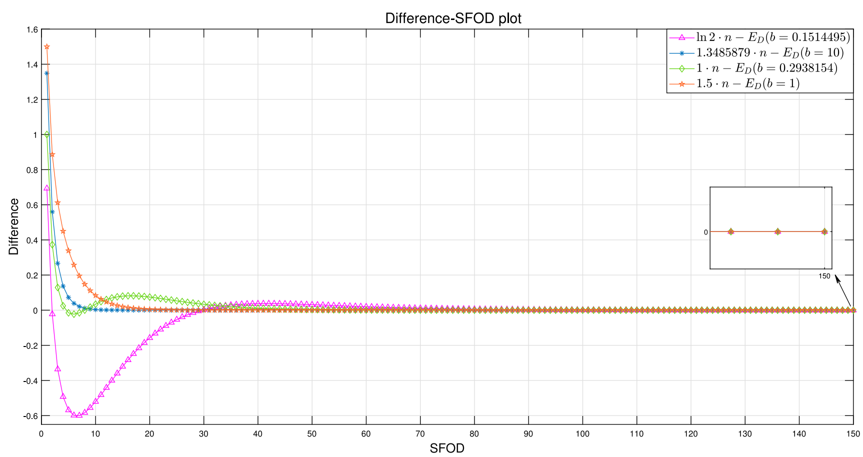

Fig. 14. The Difference-SFOD plot for the difference between the fitted values (use straight lines for fitting) and actual Deng entropy values.

k, which are used for fitting the LTDE. It is noticeable that there is poorer fitting when SFOD is relatively small, but more favorable fitting results are achieved with larger SFOD. To provide a more intuitive representation of the discrepancy between the actual Deng entropy values and their fitted values, Fig. 14 illustrates the difference between the two, namely: $k \cdot n - E_D$ plot, where n is the SFOD. It is evident that for smaller values of b and SFOD, the difference exhibits fluctuations around 0. However, as SFOD steadily increases, the difference gradually converges to 0. Consequently, it serves to validate the linearity in Deng entropy and the effectiveness of our proposed methodology.

In thermodynamics, entropy is a function that describes the thermodynamic state of a system. A specific system state can yield the entropy corresponding to that state, but a specific entropy cannot determine the system's state. Similarly, for a given FOD, there exists a one-to-many relationship between entropy and mass functions. A specific mass function corresponds to a specific entropy, but a particular entropy can correspond to multiple mass functions. Therefore, it is challenging to deduce mass functions from a particular entropy. However, in this paper, using the MFG, we demonstrate that it is possible to derive at least one mass function from a LTDE. This approach provides a new perspective for exploring unknown systems.

Linearity, owing to its simplicity, provides a significant advantage in entropy calculation within the context of the LTDE. For instance, consider a sufficiently large FOD, such as $N = 10^{10}$. According to Eq. (3), when calculating Deng entropy, it needs to compute $2^{10^{10}}$ data sets, with each data set requiring the computation of $\log_2\left[\frac{m(A)}{2^{|A|}-1}\right]$. Such calculations are exceedingly intricate and computationally demanding, posing a formidable challenge in terms of computational resources. In contrast, with the introduction of the LTDE, these calculations become remarkably simplified. Once the slope is determined, a single multiplication suffices to obtain the entropy. The linearity greatly reduces the computational burden associated with Deng entropy, offering a more efficient and manageable approach, especially when dealing with the large FOD. Here is a more specific example shown in Example 5.1.

Example 5.1. *Given a FOD $\Theta_n = \{\theta_1, \theta_2, \ldots, \theta_N\}$ and a mass function m_l defined on that FOD within a system. When $N = 10^{10}$ and $m_l(A) = \frac{8^{|A|}}{\sum_{F\subseteq\Theta} 8^{|F|}}$ for any non-empty subsets $A, F \subseteq \Theta$, measure the uncertainty in the system under this mass function.*

We can use Deng entropy to measure the uncertainty in the system. Since the cardinality of the FOD is 10^{10}, the cardinality of the power set would be $2^{10^{10}}$. According to Eq. (3), Deng entropy can be calculated as follows:

$$E_D(m_l) = -\sum_{A\in 2^\Theta} \frac{8^{|A|}}{\sum_{F\subseteq\Theta} 8^{|F|}} \cdot \log_2\left[\frac{\frac{8^{|A|}}{\sum_{F\subseteq\Theta} 8^{|F|}}}{2^{|A|}-1}\right] \tag{36}$$

The calculation of Eq. (36) faces two main challenges. The first challenge is the summation of $(2^{10^{10}}-1)$ terms, which is a large number of terms. The second challenge is the computation of $8^{10^{10}}$, which introduces substantial numerical challenges due to the exceedingly large value involved. Consequently, using Eq. (36) for this calculation would become extremely complex. Nevertheless, it is worth noting that m_l is a PAMF, and according to Theorem 3.1, the corresponding Deng entropy would be LTDE. Hence, the calculation can be simplified using Eq. (20) as follows:

$$\begin{aligned} E_D(m_l) &= \left[\log_2(b+1) - \log_2\left(\frac{b}{2}\right)\cdot\frac{b}{b+1}\right]\cdot N \\ &= \left[\log_2(8+1) - \log_2\left(\frac{8}{2}\right)\cdot\frac{8}{8+1}\right]\cdot 10^{10} \\ &\approx 1.3921\times 10^{10} \end{aligned} \tag{37}$$

Therefore, the uncertainty of the system can be rapidly computed using LTDE, significantly reducing computational complexity. LTDE proves to be a crucial computational tool, particularly when dealing with a large FOD.

The presence or absence of linearity depends on what is considered as a crucial aspect to be discussed. A essential determinant is how the information measure is defined. For instance, linearity is non-existent in Shannon entropy because the maximum Shannon entropy, as an upper bound, can only reach $\log(n)$. Therefore, a necessary condition for the existence of linearity is that the information measure, represented by entropy E with its corresponding maximum entropy E_{max}, and SFOD n, $\exists k>0$, such that $n\to\infty$, $E_{max} > k\cdot n$. Another significant determinant is how the mass function is defined. Within the framework of Deng entropy as the measure of information, consider the following two mass functions:

$$m_1(F) = \begin{cases} \frac{1}{n} & ,\ \text{if}\ \ |F| = 1 \\ 0 & ,\ \text{otherwise} \end{cases} \tag{38}$$

$$m_2(F) = \begin{cases} \frac{2}{n\cdot(n-1)} & ,\ \text{if}\ \ |F| = 2 \\ 0 & ,\ \text{otherwise} \end{cases} \tag{39}$$

According to Eq. (3), it can be calculated that:

$$E_D(m_1) = \log_2(n) \tag{40}$$

$$E_D(m_2) = \log_2\left[\frac{3n\cdot(n-1)}{2}\right] \tag{41}$$

The Entropy-SFOD plot of these mass functions is shown in Fig. 15. It is evident that when the mass function follows either Eq. (38) or Eq. (39), the corresponding Deng entropy does not exhibit linearity property. It implies that the linearity property in Deng entropy is not universal, since the definition of the mass function would influence the manifestation of linearity.

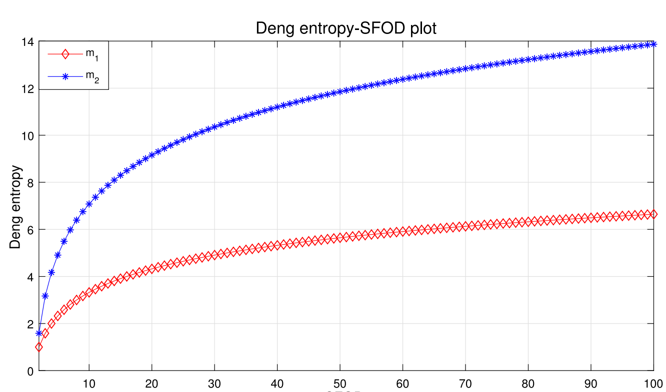

Fig. 15. The Deng entropy-SFOD plot of these two mass functions (please note that the x-axis coordinates start from 2).

What is the implicit meaning of linearity in Deng entropy? This is an open issue undoubtedly. The proposed information dimension based on Deng entropy also shows a strong linearity [57]. The slope is regarded as a dimension. It inspires us to present a hypothesis: Is there an equation relating entropy, dimension, and SFOD, which can be expressed as:

$$E = D * f(n) \tag{42}$$

where E denotes entropy, D stands for information dimension, $f(n)$ represents a function that depends on the SFOD. In Shannon entropy $f(n)=\log(n)$, while in Deng entropy $f(n)=n$. With a fixed information dimension, the entropy increases with SFOD in a manner determined by the scale associated with the information dimension. Therefore, D serves as a scale that quantifies the rate of information growth.

In conclusion, the linearity in Deng entropy remains a subject that demands further investigation. In the future work, we will conduct a more comprehensive and in-depth exploration of this topic.

6. Conclusion

Linearity plays a crucial role in various domains. Intriguingly, linearity can also be observed in Deng entropy-related research, including the maximum Deng entropy and information dimension based on Deng entropy. In this paper, we conduct an in-depth exploration of the linear relationship between Deng entropy and the scale of the frame of discernment (SFOD). The primary outcomes and contributions can be summarized as follows:

- The linearity in Deng entropy is systematically reviewed and discussed.
- The slope of Deng entropy is nothing else but the information fractal dimension of mass function.
- The linear type Deng entropy (LTDE) is defined.
- A set of specific mass functions named power assigned mass function (PAMF) which could generate LTDE are discovered.
- The mass function generator (MFG) which could derive mass functions corresponding to a LTDE with a given slope is presented.
- Some proofs, numerical examples and an error analysis are presented to validate the effectiveness of our findings.

CRediT authorship contribution statement

Tong Zhao: Conceptualization, Data curation, Formal analysis, Methodology, Resources, Software, Validation, Visualization, Writing – original draft, Writing – review & editing. **Zhen Li:** Funding acquisition, Project administration, Supervision, Writing – review & editing. **Yong Deng:** Funding acquisition, Investigation, Project administration, Resources, Supervision, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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