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Information fractal dimension of Random Permutation Set

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ABSTRACT

Random permutation set (RPS) is a recently introduced set based on the Dempster–Shafer evidence theory, which considers all possible permutations of the elements within a given set. The information dimension is a significant fractal dimension which plays a vital role in the information theory. Nevertheless, how to develop the information dimension of a specific permutation mass function in RPS remains an unresolved problem. To solve this problem, we propose a new dimension named information fractal dimension of Random Permutation Set. Moreover, several properties of the proposed dimension are explored and numerical examples are provided to illustrate its effectiveness. The research discovers an interesting property related to the permutation mass function corresponding to the maximum RPS entropy: its information dimension is 2, which is equivalent to the fractal dimension of Brownian motion and Peano curve.

1. Introduction

In the information theory, the measurement of uncertainty holds significant importance. A wide range of theories have been developed to cope with uncertainty, including probability theory [1], Dempster–Shafer evidence theory (evidence theory) [2,3], Z-numbers [4], rough sets [5], fuzzy sets [6], D numbers [7] and complex evidence theory [8–10]. These theories have found practical application in numerous fields, such as classification [11,12], decision making [13], risk assessment [14,15], pattern classification [16] and group decision making [17,18]. Entropy, as a powerful tool for measuring the uncertain degree of information, has led to the development of various related theories, including Shannon entropy [19], Rényi entropy [20], fuzzy entropy [21], Deng entropy [22], complex entropy [23], Tsallis entropy [24], and has extended to information quality [25], complex systems [26], divergence measure [27–29], network theory [30,31], mutual information matrix [32,33] and various other domains [34,35]. Of particular interest is the Deng entropy [22], which serves as an effective parameter for measuring uncertainty in evidence theory, and has thus sparked extensive attention regarding its properties [36,37], generalizations [38,39] and applications [40].

Set theory stands as a fundamental theory which describes the collections of objects or elements [41]. The majority of existing theories are grounded in set theory, including concepts like the sample space in probability theory [1] and the power set in evidence theory [2,3]. A recent study has proposed a new perspective on the power set, suggesting that it represents the complete set of combinations attainable from the elements within a frame of discernment [42]. Inspired by this interpretation, a novel set named random permutation set (RPS) is presented, which represents the complete set of permutations attainable from the elements within a frame of discernment [43]. RPS has found applications in diverse fields, including uncertainty measurement [44,45], pattern recognition [46,47] and game theory [48].

Fractal dimension is a measure used to quantify the complexity of fractal structures. It is determined by analyzing how the scale change and quantity change of a fractal image are interrelated [49]. Generally speaking, a higher fractal dimension implies a more intricate and complex structure [50,51]. Fractal dimension has found practical use in a wide range of fields associated with fractal theory, including complex networks [52,53], non-linear dynamic systems [54,55] and aggregate systems [56]. Each interesting curve has its own fractal dimension. For example, the Hausdorff dimension [57] of the trajectory of Brownian motion on a plane is equal to 2 [58], the Sierpinski triangle has a fractal dimension of $ln3/ln2$ [59,60] while the Peano curve has a fractal dimension of 2 [49]. The information dimension [61,62], a form of fractal dimension, holds significant importance in the analysis of probability distributions [63].

Recently, a new kind of dimension based on fractal theory and Deng entropy [22] has been proposed [64]. However, how to develop an information dimension of a specific permutation mass function in RPS

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remains an unresolved problem. To resolve this problem, we propose an information dimension of RPS based on RPS entropy. The proposed dimension is compatible with both Rényi information dimension [61,62] and information fractal dimension of mass function [64]. Additionally, multiple numerical examples are offered to exemplify the utilization of the proposed dimension. By analyzing these numerical examples, an intriguing discovery has been made: the information dimension of permutation mass function corresponding to the maximum RPS entropy is 2, which is equivalent to the fractal dimension of Brownian motion and Peano curve. This novel discovery establishes a link between information theory and fractal geometry, with important implications for both fields.

The subsequent sections of this article are structured as follows. Section 2 makes an introduction to the preliminary concepts and background information. Section 3 proposes the information dimension of RPS and define several distribution mode in RPS. Section 4 applies multiple numerical examples to exemplify the utilization of the proposed dimension. Section 5 offers a concise conclusion.

2. Preliminaries

How to model and measure the uncertainty and dynamics of the system have attracted a lot of attentions, with research focusing on areas such as networks [65,66], chaotic attractors [67], regret minimization [68], multi-agent learning [69] and others [70]. Some introductions to the preliminary concepts and background information is provided in this section, including RPS, RPS entropy, maximum RPS entropy and information dimension.

2.1. Random permutation set

Random permutation set (RPS) is a recently introduced set which is comprised of the permutation event space (PES) and permutation mass function (PMF) [43]. To provide an overview of RPS, we introduce some fundamental definitions below.

Definition 2.1 (*Permutation Event Space*). Consider a fixed set Ω consisting of N elements that are mutually exclusive and collectively exhaustive, denoted as $\{\omega_1, \omega_2, \dots, \omega_n\}$, the **permutation event space (PES)** is defined as the set which is comprised of all possible permutations of the elements in Ω. It can be expressed as follows:

$$\begin{aligned} PES(\Omega) &= \{M_{ij} | i = 0, 1, 2, \dots, N; j = 1, 2, \dots, P(N, i)\} \\ &= \{\emptyset, \{\omega_1\}, \dots, \{\omega_{N-1}\}, \{\omega_N\}, \{\omega_1, \omega_2\}, \{\omega_2, \omega_1\}, \dots, \{\omega_{N-1}, \\ &\quad \omega_N\}, \{\omega_N, \omega_{N-1}\}, \dots, \{\omega_1, \omega_2, \dots, \omega_N\}, \dots, \{\omega_N, \omega_{N-1}, \dots, \omega_1\}\} \end{aligned} \tag{1}$$

here, $P(N, i)$ refers to the i-permutation of N, which is defined as $P(N, i) = \frac{N!}{(N-i)!}$. In the permutation event space (PES), each permutation event M_{ij} represents a possible i-permutation of N elements in Ω. The index i represents the cardinality of M_{ij}, and the index j signifies the existence of j distinct i-permutations.

Definition 2.2 (*Random Permutation Set*). Consider a fixed set Ω consisting of N elements that are mutually exclusive and collectively exhaustive, denoted as $\{\omega_1, \omega_2, \dots, \omega_n\}$, the **random permutation set (RPS)** is a set composed of pairs of elements, which can be defined as follows:

$$RPS(\Omega) = \{\langle M, \mathscr{M}(M)\rangle | M \in PES(\Omega)\} \tag{2}$$

The **permutation mass function (PMF)**, denoted as $\mathscr{M}$, is defined by the following expression:

$$\mathscr{M} : PES(\Omega) \to [0, 1] \tag{3}$$

the constraints are $\mathscr{M}(\emptyset) = 0$ and $\sum_{M \in PES(\Omega)} \mathscr{M}(M) = 1$.

RPS exhibits compatibility with both evidence theory [2,3] and probability theory [1]. The PES of the RPS degenerates into the power set when the sequence of the elements in the permutation events are not considered, and it further degenerates into the sample space in the case where each permutation event includes only one single element. Likewise, under the same conditions, the PMF of RPS can degenerate into a BPA and a probability distribution [43].

2.2. RPS entropy and maximum RPS entropy

Recently, RPS entropy has been introduced as a measure of uncertainty for RPS [44]. Additionally, RPS maximum entropy and its corresponding PMF condition have been proposed and proofed [45].

Definition 2.3 (*RPS Entropy*). Consider a RPS denoted as $RPS(\Omega) = \{\langle M_{ij}, \mathscr{M}(M_{ij})\rangle | M_{ij} \in PES(\Omega)\}$, the RPS entropy is by the following expression:

$$H_{RPS}(\mathscr{M}) = -\sum_{i=1}^{N} \sum_{j=1}^{P(N,i)} \mathscr{M}(M_{ij}) \log\left(\frac{\mathscr{M}(M_{ij})}{F(i) - 1}\right) \tag{4}$$

here, $P(N, i)$ refers to the i-permutation of N. $F(i)$ is the sum from 0-permutation of i to i-permutation of i, which can be calculated as $F(i) = \sum_{a=0}^{i} P(i, a) = \sum_{a=0}^{i} \frac{i!}{(i-a)!}$.

RPS entropy exhibits compatibility with both Shannon entropy [19] and Deng entropy [22]. The RPS entropy degenerates into Deng entropy when the sequence of the elements in the permutation event is not considered, and it further degenerates into Shannon entropy in the case where each permutation event is constrained to include only one single element [44].

Definition 2.4 (*The PMF Condition For Maximum Entropy Of RPS*). The maximum entropy of RPS is attained only when the PMF meets the following condition:

$$\mathscr{M}(M_{ij}) = \frac{F(i) - 1}{\sum_{i=1}^{N} [P(N, i)(F(i) - 1)]} \tag{5}$$

It can be derived from this definition that there is a positive correlation between the cardinality of a certain permutation event and the magnitude of its corresponding PMF value [45].

Definition 2.5 (*The Analytic Solution For Maximum Entropy Of RPS*). The maximum entropy of a RPS can be obtained through an analytical solution, which is expressed as:

$$H_{max-RPS} = \log\left(\sum_{i=1}^{N} [P(N, i)(F(i) - 1)]\right) \tag{6}$$

2.3. Information dimension

Information dimension belongs to the family of fractal dimensions, which can be used to describe the behavior of chaotic attractors [61,62,71]. Recently, a new dimension named information fractal dimension has been proposed to deal with mass function [64]. We will briefly introduce some basic definitions about information dimension below.

Definition 2.6 (*Rényi Dimension*). Let Z be a discrete random variable and $r \in \mathbb{R}_{\geqslant 0}$, the Rényi dimension of order r is defined as follows [61,71]:

$$D_r(Z) = \lim_{a \to 0} \frac{\frac{1}{1-r} \log \sum_{i=1}^{N} P_i^r}{-\log a} \tag{7}$$

where the numerator is Rényi entropy of order r [20], P_i is the probability of the event $\{Z = z_i\}$, $N = N(a)$ is the total number of a-boxes with $P_i > 0$.

As the order of the Rényi dimension approaches 1, the denominator of $\frac{1}{1-r}$ approaches 0. This condition can be addressed by applying L'Hôpital's rule:

$$\lim_{a\to 0}\frac{\frac{1}{1-r}\log\sum_{i=1}^{N}P_i^r}{-\log a}\xrightarrow{r\to 1}\lim_{a\to 0}\frac{\sum_{i=1}^{N}P_i\log P_i}{\log a} \tag{8}$$

the 1-order Rényi dimension in Eq. (8) is defined as **information dimension** [62]:

$$D_P=\lim_{a\to 0}\frac{\sum_{i=1}^{N}P_i\log P_i}{\log a}=\lim_{n\to\infty}\frac{-\sum_{i=1}^{N}P_i\log P_i}{\log n} \tag{9}$$

where the numerator is Shannon entropy H_S [19].

Definition 2.7 (*Information Dimension Of Mass Function*)**.** For a framework of discernment Ω, the power set is $2^{\Omega}=\{M_1,M_2,\dots,M_{2^N}\}$, a mass function is $m(M)$. Its information dimension is defined as follows [64]:

$$D_m=\frac{-\sum_{M_i\in 2^{\Omega}}m(M_i)\log\left(\frac{m(M_i)}{2^{|M_i|-1}}\right)}{\log\sum_{M_i\in 2^{\Omega}}\left(2^{|M_i|}-1\right)^{m(M_i)}} \tag{10}$$

where the numerator is Deng entropy H_{DE} [22], $(2^{|M_i|}-1)$ is the size of the power set corresponding to the focal element M_i. When the mass function degenerates into probability distribution, the dimension will degenerate into Rényi information dimension [64].

3. Information dimension of RPS

The focus of this section is to introduce our newly proposed dimension specifically designed to deal with RPS. Several properties of this dimension will be discussed and proved. Additionally, some specific distribution mode of PMF in RPS are presented and assigned with respective names.

3.1. Information dimension of RPS

Definition 3.1 (*Information Fractal Dimension Of RPS*)**.** Consider a fixed set $\Omega=\{\omega_1,\omega_2,\dots,\omega_n\}$ consisting of N elements that are mutually exclusive and collectively exhaustive, its corresponding RPS is denoted as $RPS(\Omega)=\{\langle M_{ij},\mathscr{M}(M_{ij})\rangle|M_{ij}\in PES(\Omega)\}$. The information dimension of the RPS is defined as follows:

$$D_{RPS}=\frac{H_{RPS}}{\log\left(\sum_{i=1}^{N}\sum_{j=1}^{P(N,i)}Y_{ij}\right)} \tag{11}$$

where the numerator H_{RPS} is RPS entropy, and Y_{ij} in this equation is:

$$Y_{ij}=\begin{cases}[F(i)-1]^{\mathscr{M}(M_{ij})} & ,\ \text{if}\ \ \mathscr{M}(M_{ij})\neq 0\\ 0 & ,\ \text{if}\ \ \mathscr{M}(M_{ij})=0\end{cases} \tag{12}$$

when the cardinality $|\Omega|=1$, which means $RPS(\Omega)=\{\langle M_{11},1\rangle\}$, we define that the information dimension of RPS is equal to zero: $D_{RPS}=0$.

3.2. Properties of information dimension of RPS

Property 1. *In the case where the sequence of the elements in the permutation event is not considered, the information dimension of RPS D_{RPS} will degenerate into information dimension of mass function D_m.*

Proof of Property 1. In the case where the sequence of the elements in the permutation event is not considered, the permutation mass function will degenerate into a BPA [43], and the RPS entropy H_{RPS} will degenerate into Deng entropy H_{DE} [44]. In the meantime, the permutation number $P(N,i)$ will degenerate into combinatorial number $C(N,i)$, then $F(i)$ should be calculated as $F(i)=\sum_{a=0}^{i}C(i,a)=2^i$ [44]. The information dimension of RPS is calculated as follows:

$$\begin{aligned}D_{RPS}&=\frac{H_{RPS}}{\log\left(\sum_{i=1}^{N}\sum_{j=1}^{P(N,i)}Y_{ij}\right)}\\&=\frac{H_{DE}}{\log\left(\sum_{i=1}^{N}\sum_{j=1}^{C(N,i)}(F(i)-1)^{m(M_{ij})}\right)}\\&=\frac{H_{DE}}{\log\left(\sum_{i=1}^{N}\sum_{j=1}^{C(N,i)}(2^{|M_{ij}|}-1)^{m(M_{ij})}\right)}\\&=\frac{H_{DE}}{\log\sum_i\left(2^{|M_i|}-1\right)^{m(M_i)}}\\&=D_m\end{aligned} \tag{13}$$

Property 2. *In the case where each permutation event is constrained to include only one single element, the information dimension of RPS D_{RPS} will degenerate into Rényi information dimension D_P.*

Proof of Property 2. In the case where each permutation event is constrained to include only one single element, the permutation mass function will degenerate into a probability distribution [43], and the RPS entropy H_{RPS} will degenerate into Shannon entropy H_S [44]. Meanwhile the permutation number $P(N,i)$ will degenerate into a constant $P(N,1)=N$, then $F(i)$ will become a constant $F(1)=2$ [44]. The information dimension of RPS is calculated as follows:

$$\begin{aligned}D_{RPS}&=\frac{H_{RPS}}{\log\left(\sum_{i=1}^{N}\sum_{j=1}^{P(N,i)}Y_{ij}\right)}\\&=\frac{H_S}{\log\left(\sum_{j=1}^{N}(F(1)-1)^{p_j}\right)}\\&=\frac{H_S}{\log N}\\&=D_P\end{aligned} \tag{14}$$

3.3. Some specific distribution mode of PMF in RPS

For the convenience of later discussion, we will define some special distribution of PMF in RPS.

Definition 3.2 (*Exclusive Distribution For A Maximum Subset*)**.** Consider a RPS: $RPS(\Omega)=\{\langle M_{ij},\mathscr{M}(M_{ij})\rangle|M_{ij}\in PES(\Omega)\}$, exclusive distribution for a maximum subset is defined as follows:

$$\mathscr{M}(M_{ij})=\begin{cases}1 & ,\ \text{if}\ \ i=N\ \text{and}\ j=k\\ 0 & ,\ \text{otherwise}\end{cases} \tag{15}$$

where N in M_{Nk} means the cardinality $|M_{Nk}|=N$, k is a fixed constant which takes values from 1 to $P(N,N)$. This mode means that all the probability is distributed to a single maximum subset of cardinality N.

Definition 3.3 (*Average Distribution For All Maximum Subsets*)**.** Consider a RPS: $RPS(\Omega)=\{\langle M_{ij},\mathscr{M}(M_{ij})\rangle|M_{ij}\in PES(\Omega)\}$, average distribution for all maximum subsets is defined as follows:

$$\mathscr{M}(M_{ij})=\begin{cases}\frac{1}{P(N,N)} & ,\ \text{if}\ \ i=N\\ 0 & ,\ \text{otherwise}\end{cases} \tag{16}$$

where N in M_{Nj} means the cardinality $|M_{Nj}|=N$. This mode means that the probability is distributed among all maximum subsets of cardinality N on average.

Definition 3.4 (*Average Distribution For All Subsets*)**.** Consider a RPS: $RPS(\Omega)=\{\langle M_{ij},\mathscr{M}(M_{ij})\rangle|M_{ij}\in PES(\Omega)\}$, average distribution for all subsets is defined as follows:

$$\mathscr{M}(M_{ij})=\frac{1}{F(N)-1}\ ,\ \text{if}\ \ i\neq 0 \tag{17}$$

where $F(N)=\sum_{a=0}^{N}P(N,a)=\sum_{a=0}^{N}\frac{N!}{(N-a)!}$ is the sum from 0-permutation of N to N-permutation of N. This mode means that the probability is distributed among all subsets excluding the empty set on average.

Definition 3.5 (*Maximum RPS Entropy Distribution*). Consider a RPS denoted as $RPS(\Omega)=\{\langle M_{ij},\mathscr{M}(M_{ij})\rangle|M_{ij}\in PES(\Omega)\}$, maximum RPS entropy distribution is defined as follows:

$$\mathscr{M}(M_{ij})=\frac{F(i)-1}{\sum_{i=1}^{N}[P(N,i)(F(i)-1)]}\ ,\ \text{if}\ \ i\neq 0 \tag{18}$$

The RPS entropy based on this distribution mode is the maximum RPS entropy.

Property 3. *Consider a fixed set $\Omega=\{\omega_1,\omega_2,\dots,\omega_n\}$, as the cardinality $|\Omega|$ approaches infinity, the infinite maximum RPS entropy distribution on the corresponding RPS is equivalent to average distribution for all maximum subsets.*

Proof of Property 3. We already know $F(i)=\sum_{a=0}^{i}\frac{i!}{(i-a)!}$ and $P(N,i)=\frac{N!}{(N-i)!}$. Assume $P(i,i)=x$, then:

$$F(i)=x+\frac{x}{1!}+\frac{x}{2!}+\cdots+\frac{x}{i!} \tag{19}$$

When $i\to\infty$, according to the power series of e^x, we can derive that:

$$\lim_{i\to\infty}F(i)=x(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{i!})=ex \tag{20}$$

Consider a fixed set $\Omega=\{\omega_1,\omega_2,\dots,\omega_n\}$, as the cardinality $|\Omega|$ approaches infinity (i.e., $|\Omega|=N\to\infty$), consider the maximum RPS entropy distribution like Eq. (18), assume $P(N,N)=N!=y$, then:

$$\begin{aligned}\lim_{N\to\infty}\mathscr{M}(M_{Nj})&=\frac{F(N)-1}{\sum_{i=1}^{N}[P(N,i)(F(i)-1)]}\\&=\frac{ey-1}{y\cdot(F(N)-1)+y\cdot(F(N-1)-1)+\cdots}\\&=\frac{ey-1}{y\cdot(ey-1)+y\cdot(\frac{ey}{N}-1)+\cdots}\\&=\frac{1}{y}\end{aligned} \tag{21}$$

Thus $P(N,N)\cdot\mathscr{M}(M_{ij})=y\cdot\frac{1}{y}=1$. This implies that when $i<N$:

$$\lim_{N\to\infty}\mathscr{M}(M_{ij})=\frac{F(i)-1}{\sum_{i=1}^{N}[P(N,i)(F(i)-1)]}=0 \tag{22}$$

This condition is equivalent to Eq. (16), which is the definition of the average distribution for all maximum subsets.

4. Numerical examples and discussions

This section offers multiple numerical examples to exemplify the utilization of the proposed information dimension of RPS. Since both the numerator and denominator in the expression of the proposed information dimension contain logarithms, using any base for calculations will yield the same final result (but note that the numerator and denominator should use the same base). For convenience, the logarithm with base 2 is used for calculations, which means the symbol log used in the paper omits the base 2 and should be interpreted as $\log_2$.

It should be noted that for a given RPS, its corresponding information dimension is a fixed constant. For a given distribution mode, as the cardinality of the set Ω gradually increases, if the corresponding information dimension converges to a constant, then the constant will be defined as the information dimension of this distribution mode.

Example 4.1. *Consider a fixed set of $\Omega_1=\{A,B\}$, a RPS defined on Ω_1 is given as follows:*

$RPS(\Omega_1)=\{\langle\{A\},0.2\rangle,\langle\{B\},0.2\rangle,\langle\{A,B\},0.3\rangle,\langle\{B,A\},0.3\rangle\}.$

Table 1
The convergence process of the distribution mode in Example 4.2.

$\lvert\Omega_n\rvert$	H_{RPS}	$\log\left(\sum_{i=1}^{N}\sum_{j=1}^{P(N,i)}Y_{ij}\right)$	D_{RPS}
1	0	0	0
2	2.0000	2.0000	1
3	3.9069	3.9069	1
4	6.0000	6.0000	1
5	8.3443	8.3443	1
6	10.9337	10.9337	1
7	13.7418	13.7418	1
8	16.7419	16.7419	1
⋮	⋮	⋮	⋮
79	389.9476	389.9476	1
80	396.2696	396.2696	1

Based on Eq. (11), we can calculate its information dimension as follows:

$$\begin{aligned}D_{RPS}&=\frac{-0.2\log\left(\frac{0.2}{2-1}\right)-0.2\log\left(\frac{0.2}{2-1}\right)-0.3\log\left(\frac{0.3}{5-1}\right)-0.3\log\left(\frac{0.3}{5-1}\right)}{\log\left[(2-1)^{0.2}+(2-1)^{0.2}+(5-1)^{0.3}+(5-1)^{0.3}\right]}\\&\approx 1.3604\end{aligned} \tag{23}$$

In the case where the sequence of the elements in the permutation event is not considered, the RPS can be rewritten as follows:

$RPS'(\Omega_1)=\{\langle\{A\},0.2\rangle,\langle\{B\},0.2\rangle,\langle\{A,B\},0.6\rangle\}.$

according to the degeneracy property, the information dimension can be calculated as follows:

$$\begin{aligned}D_{RPS'}&=\frac{-0.2\log\left(\frac{0.2}{2-1}\right)-0.2\log\left(\frac{0.2}{2-1}\right)-0.6\log\left(\frac{0.6}{4-1}\right)}{\log\left[(2-1)^{0.2}+(2-1)^{0.2}+(4-1)^{0.6}\right]}\\&\approx 1.1752\\&=D_m\end{aligned} \tag{24}$$

In the case where each permutation event is constrained to include only one single element, the RPS can be rewritten as follows:

$RPS''(\Omega_1)=\{\langle\{A\},0.5\rangle,\langle\{B\},0.5\rangle\}.$

according to the degeneracy property, the information dimension can be calculated as follows:

$$\begin{aligned}D_{RPS''}&=\frac{-0.5\log\left(\frac{0.5}{2-1}\right)-0.5\log\left(\frac{0.5}{2-1}\right)}{\log\left[(2-1)^{0.5}+(2-1)^{0.5}\right]}\\&=1\\&=D_S\end{aligned} \tag{25}$$

The calculation results in Example 4.1 are consistent with Property 1 and Property 2 proposed in Section 3, indicating the correctness and effectiveness of the proposed properties.

Example 4.2. *Consider a fixed distribution mode:* ***exclusive distribution for a maximum subset****, consider a variable set $\Omega_n=\{\omega_1,\omega_2,\dots,\omega_n\}$ with its cardinality gradually increasing, where $|\Omega_n|=N$ is an integer variable that increases from 1. The corresponding RPS is: $RPS(\Omega_n)=\{\langle\{\omega_1,\omega_2,\dots,\omega_n\},1\rangle\}$.*

The results are presented in Table 1, where the first column corresponds to the cardinality of the set Ω_n, the second column shows the RPS entropy, the third column represents the denominator in the expression for information dimension, and the last column displays the information dimension of RPS.

Note that the x-axis in Fig. 1 represents the denominator of the expression for information dimension, while the y-axis shows the RPS entropy. This implies that the slope of the line corresponds to the information dimension. We will use this plotting method in the following examples as well (see Figs. 2, 3 and 6).

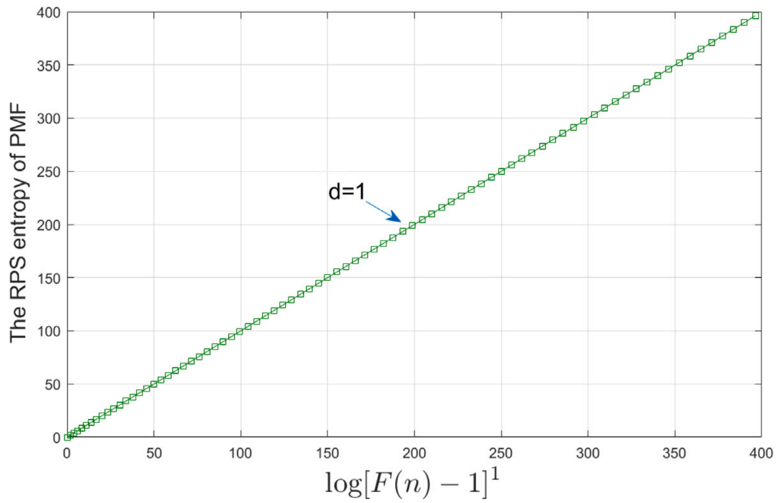

Fig. 1. The result of Example 4.2, where $|\Omega_n| = 1, 2, \ldots, 80$.

Table 2
The convergence process of the distribution mode in Example 4.3.

$\|\Omega_n\|$	H_{RPS}	$\log\left(\sum_{i=1}^{N}\sum_{j=1}^{P(N,i)} Y_{ij}\right)$	D_{RPS}
1	0	0	0
2	3.0000	2.0000	1.5000
3	6.4919	3.2361	2.0061
4	10.5850	4.8350	2.1893
5	15.2512	6.9764	2.1861
6	20.4255	9.5070	2.1485
7	26.0410	12.3019	2.1168
8	32.0411	15.2996	2.0942
⋮	⋮	⋮	⋮
79	778.4526	388.5049	2.00371
80	791.0964	394.8269	2.00365
⋮	⋮	⋮	⋮
300	4083.9980	2041.2777	2.00071
500	7536.1497	3767.3535	2.00038
700	11225.4682	5612.0128	2.00026

Regarding the distribution mode in Example 4.2, as the cardinality of Ω_n increases, the RPS information dimension for this mode of distribution converges to 1. As shown in Fig. 1, the curve exhibits a linear relationship with a slope of 1, indicating that the RPS information dimension for this distribution mode is 1.

Example 4.3. *Consider a fixed distribution mode:* ***average distribution for all maximum subsets****, consider a variable set* $\Omega_n = \{\omega_1, \omega_2, \ldots, \omega_n\}$ *with its cardinality gradually increasing, where* $|\Omega_n| = N$ *is an integer variable that increases from 1. The corresponding RPS is:* $RPS'(\Omega_n) = \{\langle\{\omega_1, \omega_2, \ldots, \omega_n\}, \frac{1}{P(N,N)}\rangle, \ldots, \langle\{\omega_n, \omega_{n-1}, \ldots, \omega_1\}, \frac{1}{P(N,N)}\rangle\}$.

The results is presented in Table 2. It can be observed that, unlike the convergence process in other examples, the RPS information dimension in Example 4.3 shows a trend of increasing and then decreasing, and gradually converges to 2 as $|\Omega_n|$ increases. To enhance the illustration of the convergence of the distribution mode, the relevant parameters for $|\Omega_n| = 300, 500, 700$ are calculated. Moreover, a mathematical proof is presented.

Proof for the convergence of the distribution mode in Example 4.3. When $N = |\Omega_n| \to \infty$, assume that $P(N,N) = N! = x$, then:

$$\lim_{N\to\infty} D_{RPS} = \frac{-\sum_{i=1}^{N}\sum_{j=1}^{P(N,i)} \mathcal{M}(M_{ij}) \log\left(\frac{\mathcal{M}(M_{ij})}{F(i)-1}\right)}{\log\left(\sum_{i=1}^{N}\sum_{j=1}^{P(N,i)} Y_{ij}\right)}$$

$$= \frac{-x \cdot \frac{1}{x} \cdot \log\left(\frac{1/x}{ex-1}\right)}{\log\left[x \cdot (ex-1)^{\frac{1}{x}}\right]}$$

$$= \frac{\log(ex^2 - x)}{\log(x) + \frac{1}{x}\log(ex-1)}$$

$$= \frac{\log(ex^2 - x)}{\log(x)}$$

$$\overset{\frac{\infty}{\infty}}{=} \frac{\frac{2ex-1}{ex^2-x}}{\frac{1}{x}}$$

$$= \frac{2ex^2 - x}{ex^2 - x}$$

$$= 2 \tag{26}$$

It proves that as $|\Omega_n| \to \infty$, the information dimension converges to 2. In other words, the information dimension of average distribution for all maximum subsets is 2.

The reasonable interpretation of the information dimension is still an open question for exploration. In spatial geometry, one dimension represents a straight line, two dimensions represent a plane, and three dimensions represent a volume. It can be observed that as the dimension increases, the complexity of the space and the information it contains also increase. By analogy, the information dimension can possibly be interpreted as a measure of information complexity. Since the numerator of the dimension function in Eq. (11) is RPS entropy, information dimension can also measure the uncertainty of the information.

In evidence theory, $m(\Omega) = 1$ denotes complete uncertainty regarding a frame of discernment, where the assignment of probabilities is unknown [2,3]. By comparing Example 4.2 and Example 4.3, we observe that the information dimension of the **average distribution for all maximum subsets** is larger, indicating that the distribution mode is more complex and uncertain than the **exclusive distribution for a maximum subset**. Thus, in RPS, the distribution mode that corresponds to the scenario where the information is entirely unknown should be the **average distribution for all maximum subsets**. This conclusion is reasonable since in the **exclusive distribution for a maximum subset** distribution mode, despite the lack of information about the individual permutation events, information about the order of the elements can be obtained. However, in the **average distribution for all maximum subsets** mode, the probability is averagely distributed among each maximum subset, resulting in the absence of any information about the order of the elements.

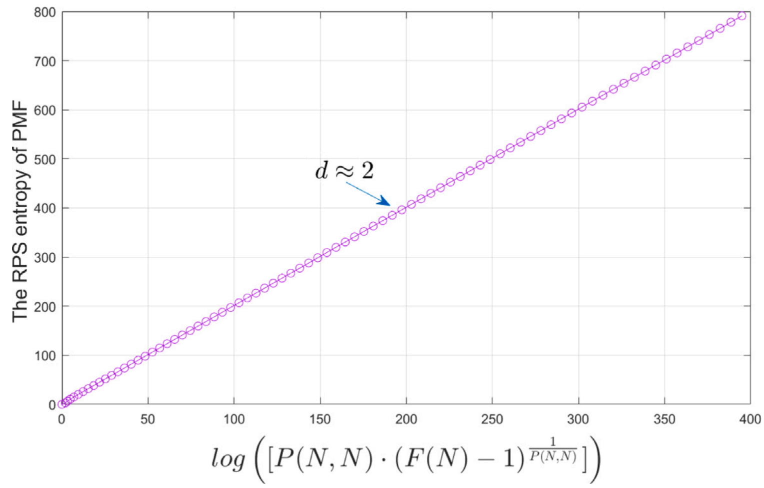

Fig. 2. The result of Example 4.3, where $|\Omega_n| = 1, 2, \ldots, 80$.

Table 3
The convergence process of the distribution mode in Example 4.4.

$\|\Omega_n\|$	H_{RPS}	$\log\left(\sum_{i=1}^{N}\sum_{j=1}^{P(N,i)} Y_{ij}\right)$	D_{RPS}
1	0	0	0
2	3.3219	2.4548	1.3533
3	6.8704	4.1523	1.6546
4	10.9278	6.0886	1.7948
5	15.5406	8.3687	1.8570
6	20.6691	10.9391	1.8895
7	26.2495	13.7428	1.9101
8	32.2231	16.7420	1.9247
⋮	⋮	⋮	⋮
79	778.4708	389.9476	1.99635
80	791.1144	396.2696	1.99640
⋮	⋮	⋮	⋮
300	4084.0028	2042.7203	1.99930
500	7536.1526	3768.7962	1.99962
700	11225.4703	5613.4555	1.99974

Table 4
The convergence process of the distribution mode in Example 4.5.

$\|\Omega_n\|$	H_{RPS}	$\log\left(\sum_{i=1}^{N}\sum_{j=1}^{P(N,i)} Y_{ij}\right)$	D_{RPS}
1	0	0	0
2	3.0000	2.2716	1.3207
3	6.2696	4.0676	1.5413
4	10.0901	6.0642	1.6639
5	14.4850	8.3632	1.7320
6	19.4045	10.9380	1.7740
7	24.7890	13.7426	1.8038
8	30.5841	16.7420	1.8268
⋮	⋮	⋮	⋮
79	773.6007	389.9476	1.98386
80	786.2263	396.2696	1.98407
⋮	⋮	⋮	⋮
300	4077.2143	2042.7203	1.99597
500	7528.6281	3768.7962	1.99762
700	11217.4608	5613.4555	1.99832

Example 4.4. *Consider a fixed distribution mode:* ***maximum RPS entropy distribution****, consider a variable set $\Omega_n = \{\omega_1, \omega_2, \ldots, \omega_n\}$ with its cardinality gradually increasing, where $|\Omega_n| = N$ is an integer variable that increases from 1. The corresponding RPS is $RPS''(\Omega_n) = \{\langle \omega_1, \frac{F(1)-1}{\sum_{i=1}^{N}[P(N,i)(F(i)-1)]}\rangle, \langle \omega_2, \frac{F(1)-1}{\sum_{i=1}^{N}[P(N,i)(F(i)-1)]}\rangle, \ldots, \langle \{\omega_n, \omega_{n-1}, \ldots, \omega_1\}, \frac{F(N)-1}{\sum_{i=1}^{N}[P(N,i)(F(i)-1)]}\rangle\}$.*

Combining the data in Fig. 3 and Table 3, we observe that the information dimension of the **maximum RPS entropy distribution** is 2, which is the same as the information dimension of the **average distribution for all maximum subsets** in Example 4.3. This result is in line with Property 3, where it is noted that the two distribution modes become equivalent when the cardinality of Ω_n approaches infinity. For instance, for $|\Omega_n| = 700$, it can be computed that $P(700, 700) \cdot \frac{F(700)-1}{\sum_{i=1}^{700}[P(700,i)(F(i)-1)]} \approx 0.99857$, which suggests that the **maximum RPS entropy distribution** has nearly distributed all probability among the maximum subsets on average.

An interesting phenomenon is that the information dimension of the **maximum RPS entropy distribution** is 2, which is equal to the fractal dimension of the Peano curve [49] and Brownian motion trajectory on a plane [58]. The images are shown in Figs. 4 and 5. This consistency establishes a connection between information theory and fractal geometry, and inspires us to consider the application of fractal concepts in solving information problems.

Example 4.5. *Consider a fixed distribution mode:* ***average distribution for all subsets****, consider a variable set $\Omega_n = \{\omega_1, \omega_2, \ldots, \omega_n\}$ with its cardinality gradually increasing, where $|\Omega_n| = N$ is an integer variable that increases from 1. The corresponding RPS is $RPS'''(\Omega_n) = \{\langle \omega_1, \frac{1}{F(N)-1}\rangle, \langle \omega_2, \frac{1}{F(N)-1}\rangle, \ldots, \langle \{\omega_n, \omega_{n-1}, \ldots, \omega_1\}, \frac{1}{F(N)-1}\rangle\}$.*

In this example, the information dimension computed from the first 80 data points is approximately 1.98, but as $|\Omega_n|$ increases, the information dimension gradually approaches 2. It can be seen from Table 4 that when $|\Omega_n| = 700$, the dimension is 1.99832, which is close to 2. Therefore, when $|\Omega_n| \to \infty$, it can be considered that the information dimension of this example's distribution mode is 2 (see Fig. 6 and Table 4).

The information dimension of RPS, as a parameter measuring the complexity of PMF distribution mode of RPS, still has an open issue regarding its specific application scenarios. Here, a possible application is proposed.

On the battlefield, there are numerous types of enemy attacks that must be responded to, which can be broadly categorized as biochemical, electromagnetic, and physical attacks. Our military has developed several general response strategies for different attack categories, and the exact response strategies depends on specific type of the attack. This can be represented by a set $\Omega = \{B_1, B_2, \ldots, B_n, E_1, E_2, \ldots, E_n, P_1, P_2, \ldots, P_n\}$, where B represents biochemical attacks, E represents electromagnetic attacks, and P represents physical attacks.

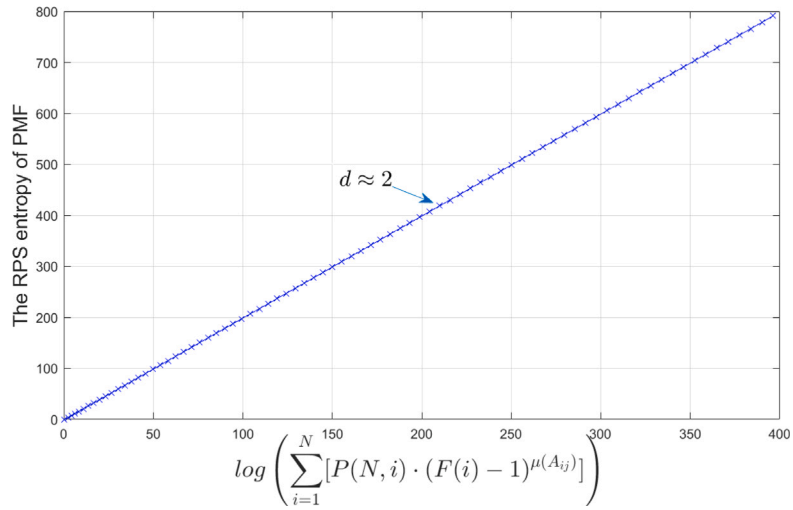

Fig. 3. The result of Example 4.4, where $|\Omega_n| = 1, 2, \dots, 80$.

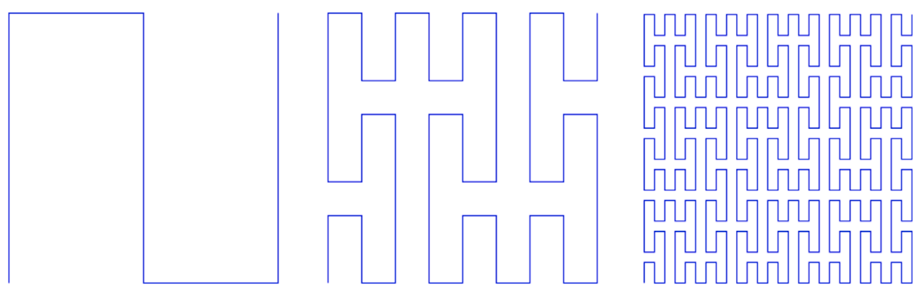

Fig. 4. The image of the Peano curve.
Source: Taken from Wikipedia.

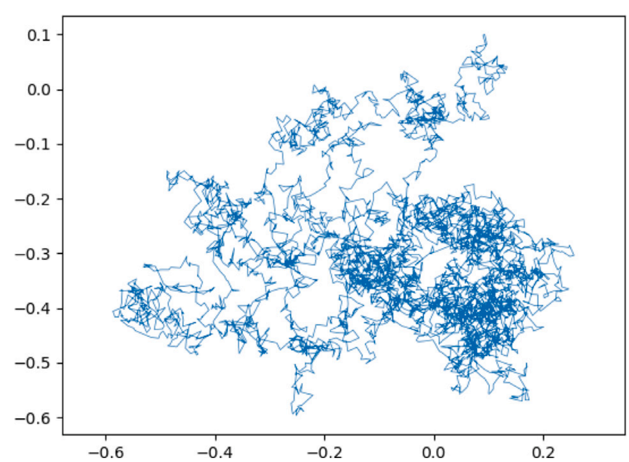

Fig. 5. The plot of Brownian motion trajectory on a plane generated by Python.

When the PMF of a specific attack is obtained through a radar, the information dimension of such a message can be measured to determine its complexity and further adjust the resorted defense strategy. If we need to quickly defend against an attack with a high information dimension, the RPS can be reduced to a set containing only B, E, and P elements, thereby significantly improving data processing efficiency. Conversely, if we need to make precise defenses against an attack with a low information dimension, the RPS can be increased to a set containing more elements to improve the specificity and accuracy of the data. This concept is similar to the notion of fractals, where the focus can be on the global structure of a shape, or on the fine-scale details, depending on the type of information required.

A recent study has shown that the maximum nonsymmetric entropy principle is of potential value for the study of complexity in a cellular automaton system [72]. Our proposed information dimension also holds promising applications in quantifying the information complexity of cellular automaton systems. In future work, we will explore various potential applications of the proposed information dimension.

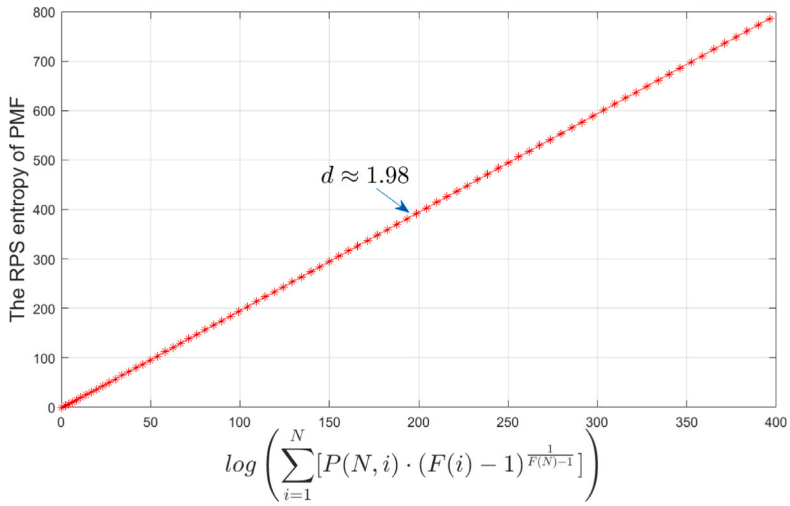

Fig. 6. The result of Example 4.5, where $|\Omega_n| = 1, 2, \ldots, 80$.

5. Conclusion

Information dimension holds significant importance in the information theory. Rényi information dimension can be used to cope with probability distribution while information fractal dimension can be applied to handle mass function in evidence theory. Recently, a novel set named random permutation set (RPS) is introduced, which considers all possible permutations of elements within a given set. Nevertheless, how to determine the information dimension of RPS is still an unresolved problem. The key contribution of our work is to develop an information dimension suitable for RPS, which ensures the compatibility with both Rényi information dimension and information fractal dimension of mass function. Several numerical examples are used to exemplify the utilization of the proposed dimension. Based on the numerical examples, an intriguing discovery has been made: the information dimension of permutation mass function corresponding to the maximum RPS entropy is 2, which is equivalent to the fractal dimension of Brownian motion and Peano curve.

CRediT authorship contribution statement

Tong Zhao: Conceptualization, Methodology, Formal analysis, Software, Writing – original draft, Visualization, Writing – review & editing. **Zhen Li:** Writing – review & editing, Supervision, Project administration, Funding acquisition. **Yong Deng:** Resources, Supervision, Funding acquisition, Writing – review & editing, Project administration.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article

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